MA 4633/6633 Practice Exam 2 Solutions

1.) pf: Let $y = x_0 + ah$, then as $h \to 0$, $y \to x_0$. Moreover $h = \frac{y - x_0}{a}$ for $a \neq 0$. Then

$$\lim_{h\to 0} \frac{f(x_0+ah) - f(x_0)}{h} = a \lim_{y \to x_0} \frac{f(y) - f(x_0)}{y - x_0} = a f'(x_0)$$ by def. of f'. if $a = 0$ then

$$\lim_{h\to 0} \frac{f(x_0+ah) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) - f(x_0)}{h} = 0 = 0 \cdot f'(x_0) \quad \square$$

2.) pf: Consider $\frac{f(x) - f(0)}{x - 0} = \frac{x^n g(x)}{x} = x^{n-1} g(x)$. if $n = 1$ then $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} g(x) = g(0)$ since g is cont. at $x = 0$. OTH if $n > 1$ then $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x^{n-1} g(x) = 0$ as g cont. at $x = 0$.
$\therefore$ f is diff. at $x = 0$ for $n \geq 1$ and $f'(0) = \begin{cases} g(0) & n = 1 \\ 0 & n > 1. \end{cases}$ $\square$

3.) pf: 1st by FTOC $\int_a^b t^x \, dt = \frac{t^{x+1}}{x+1}$ for $x \neq -1$. $\therefore$ $f(x) = \begin{cases} \frac{b^{x+1}}{x+1} - \frac{a^{x+1}}{x+1} & x \neq -1 \\ \log b - \log a & x = -1. \end{cases}$

Study $\lim_{x \to -1} \frac{b^{x+1} - a^{x+1}}{x+1}$ this is an indeterminate form so by L'Hôpital, $\lim_{x\to -1} \frac{b^{x+1} - a^{x+1}}{x+1} = \lim_{x \to -1} \frac{b^{x+1} \log b - a^{x+1} \log a}{1}$
$= \log b - \log a = f(-1)$. $\therefore$ $f(x) \to f(-1)$ as $x \to -1$. thus f is cont. at $x = -1$. $\square$

4.) pf: 1st notice $\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{\sin x} = \lim_{x \to 0} \left(\frac{x}{\sin x}\right) \cdot x \sin(\frac{1}{x})$. Know from prev. prop. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
and $\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$ by squeeze thm. $\therefore$ $\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} x \sin(\frac{1}{x})\right) = 1 \cdot 0 = 0$

Next, $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ $\forall x \neq 0$, and $g'(x) = \cos x$. Notice $\forall \delta > 0$ if $x \in (-\delta, \delta)$ then $\left|\frac{1}{\cos x}\right| \leq M$ for some $M > 0$. $\cancel{\lim_{x\to 0} \frac{2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})}{\cos x} = 0}$ so $\lim_{x\to 0} \frac{\cos(\frac{1}{x})}{\cos x}$ DNE as $\left|\frac{\cos(\frac{1}{x})}{\cos x}\right| \leq M \cos(\frac{1}{x})|$
$\therefore$ $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ DNE. $\square$

5.) pf: Since $\int_a^b f = 0 \Rightarrow U(f,P) = L(f,P) = 0$ $\forall$ partition $P = \{a = x_0 < \cdots < x_n = b\}$. OTH. $U(f,P), L(f,P) \geq 0$ since $f(x) \geq 0$ $\forall x \in [a,b]$. But $U(f,P) = \sum_{j=1}^{n} \left(\sup_{x_{j-1} \leq x \leq x_j} f(x)\right)(x_j - x_{j-1})$ & $L(f,P) = \sum_{j=1}^{n} \left(\inf_{x_{j-1} \leq x \leq x_j} f(x)\right)(x_j - x_{j-1})$
$\therefore$ $\sup_{x_{j-1} \leq x \leq x_j} f(x) = \inf_{x_{j-1} \leq x \leq x_j} f(x) = 0$. $\therefore$ $f(x) = 0$ on any partition P. $\therefore$ $f(x) = 0$ on any dense subset
since f is cont. $\Rightarrow$ $f = 0$ on $[a,b]$. $\square$

6.) pf: Consider $\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{j=1}^{n} f(\frac{j}{n})\right| = \left|\int_0^1 \left(f(x) - \frac{1}{n}\sum_{j=1}^{n} f(\frac{j}{n})\right) dx\right| = \left|\int_0^1 \frac{1}{n}\sum_{j=1}^{n} \left(f(x) - f(\frac{j}{n})\right) dx\right| = (*)$

Since f is Lipschitz $\exists L$ s.t. $|f(x) - f(y)| \leq L|x - y|$ $\forall x,y \in [0,1]$. $\therefore$ $(*) \leq \left|\frac{L}{n} \int_0^1 \sum_{j=1}^{n} (x - \frac{j}{n})\, dx\right|$
$= \left|\frac{L}{n}\int_0^1 \left(nx - \frac{n(n+1)}{2n}\right) dx\right| = \frac{L}{n}\left|\left(\frac{nx^2}{2} - \frac{(n+1)}{2}x\right)\Big|_0^1\right| \leq \frac{L}{n}$. $\square$

7.) pf:

S'pose such an E exists. Then define $F(a,b)=\int_a^b \chi_E$ $\therefore$ $F(a,b)=\frac{b-a}{2}$ by assumption. Then by FToC, $F_a = -\chi_E(a)$ and $F_b = \chi_E(b)$. OTOH $F_a = -\frac{1}{2}$ and $F_b = \frac{1}{2}$ $\therefore$ by partial deriv must have $\chi_E(b)=\frac{1}{2}$ and $\chi_E(a)=\frac{1}{2}$ but $\chi_E(a)=\begin{cases} 1 & a\in E \\ 0 & a\notin E\end{cases}$ so $\chi_E(a)=\frac{1}{2}$ is never true. Similarly for $\chi_E(b)$. $\therefore$ no such E exists. $\square$

8.) pf:

Define $F(a,b)=\int_a^{\frac{a+b}{2}} f$, thus by assumption $F(a,b)=\int_{\frac{a+b}{2}}^b f$. By FToC and chain rule have $F_a = \frac{1}{2} f(\frac{a+b}{2}) - f(a)$, $F_b = \frac{1}{2} f(\frac{a+b}{2})$. OTOH $F_a = -\frac{1}{2} f(\frac{a+b}{2})$ and $F_b = f(b) - \frac{1}{2} f(\frac{a+b}{2})$. Comparing terms have: $\frac{1}{2} f(\frac{a+b}{2}) - f(a) = -\frac{1}{2} f(\frac{a+b}{2}) \Rightarrow f(a) = f(\frac{a+b}{2})$. Similarly $f(b) = f(\frac{a+b}{2})$ $\therefore$ $f(a)=f(b)$ $\forall\, 0\le a<b\le 1$, thus f is const. on $[0,1]$. $\square$

9.) pf:

Let E be any countable subset in $\mathbb{R}$. $\therefore$ E can be enumerated i.e. $E=\{x_n : n\in\mathbb{N}\}$. Now cover E by $\{(x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n})\}_{n=1}^{\infty}$ i.e. $E \subseteq \bigcup_{n=1}^{\infty} (x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n})$. Notice $\sum_{n=1}^{\infty} \ell(x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n}) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$. $\therefore$ E has measure zero. $\square$